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Dual equations and solutions of I-type crack of dynamic problems in piezoelectric materials *

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(Contributed by WANG Biao)

Abstract This paper firstly works out basic differential equations of piezoelectric materials expressed in terms of potential functions, which are introduced in the very beginning. These equations are primarily solved through Laplace transformation, semi-infinite Fourier sine transformation and cosine transformation. Secondly, dual equations of dynamic cracks problem in 2D piezoelectric materials are established with the help of Fourier reverse transformation and the introduction of boundary conditions. Finally, according to the character of the Bessel function and by making full use of the Abel integral equation and its reverse transform, the dual equations are changed into the second type of Fredholm integral equations. The investigation indicates that the study approach taken is feasible and has potential to be an effective method to do research on issues of this kind.

Key words piezoelectric material, dynamic crack, potential function, coupled integral equation, integral transformation

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Introduction

Notable achievements have been obtained in the field of static damage and fracture behavior of piezoelectric materials, which is easily concluded from representational investigations made by Wang[1,2], Zhou[3], and Pak[4], et al. With more and more interest shown in this issue, the dynamic fracture analysis of the piezoelectric medium has become a newly rising field. Khutoryansky and Sosa[5], took the lead in providing the governing equations and basic solutions to the electro-elastic transient problem in the piezoelectric material. Hou[6] and Chen[7], et al. investigated the transient response to anti-plane cracks in the piezoelectric medium. However, for 3D or plane dynamic fracture mechanics problems in piezoelectric materials, the decoupled analysis of the governing equations cannot be carried out because of the anisotropy of the materials. Consequently, by introducing functions of displacement and electrical potential, this paper investigates basic equations and dual equations of 2D dynamic problems in piezoelectric materials and, in particular, explores several skills in working out their solutions.

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1 Expression of dynamic equation and potential function in piezoelectric materials

It is assumed that plane xOy is anisotropic and the inertia force is considered without any attention paid to the effect of body force and body charge. The constitutive equation of piezoelectrics can be written as

$$\sigma_{ij} = c_{ijkl}\gamma_{kl} - e_{kij}E_k, \tag{1}$$

$$D_i = e_{ikl}\gamma_{kl} + \varepsilon_{ik}E_k, \tag{2}$$

where c_{ij}, e_{ij} and ε_{ij} are elastic constants, piezoelectric constants, and dielectric constants, respectively; while $\sigma_{ij}, \gamma_{ij}, D_i$ and E_i are the components of stress, strain, electric displacement and electric field, respecitvely. The geometric equation is

$$\gamma_{ij} = (u_{j,i} + u_{i,j})/2, \quad E_i = -\varphi_{,i}, \tag{3}$$

where $u_i = u_i(x, y, t)$ is the component of displacement functions; $\varphi = \varphi(x, y, t)$ is the electric potential function. They are the function of x, y and t. The governing equation is

$$\sigma_{ji,i} = \rho\ddot{u}_j, \quad D_{j,j} = 0. \tag{4}$$

Substituting Eqs.(1)–(3) into Eq.(4), the governing equations are obtained as

$$\begin{cases} c_{11}\dfrac{\partial^2 u_x}{\partial x^2} + c_{44}\dfrac{\partial^2 u_x}{\partial y^2} + (c_{13} + c_{44})\dfrac{\partial^2 u_y}{\partial x \partial y} + (e_{15} + e_{31})\dfrac{\partial^2 \varphi}{\partial x \partial y} = \rho\ddot{u}_x, \\ c_{33}\dfrac{\partial^2 u_y}{\partial y^2} + c_{44}\dfrac{\partial^2 u_y}{\partial x^2} + (c_{13} + c_{44})\dfrac{\partial^2 u_x}{\partial x \partial y} + e_{15}\dfrac{\partial^2 \varphi}{\partial x^2} + e_{33}\dfrac{\partial^2 \varphi}{\partial y^2} = \rho\ddot{u}_y, \\ e_{33}\dfrac{\partial^2 u_y}{\partial y^2} + e_{15}\dfrac{\partial^2 u_y}{\partial x^2} + (e_{15} + e_{31})\dfrac{\partial^2 u_x}{\partial x \partial y} - \varepsilon_{11}\dfrac{\partial^2 \varphi}{\partial x^2} - \varepsilon_{33}\dfrac{\partial^2 \varphi}{\partial y^2} = 0. \end{cases} \tag{5}$$

The equations above are the coupled field ones of the 2D problem in isotropic piezoelectric materials, in which the elastic displacement component $u_j = u_j(x, y, t)$ and electric potential $\varphi = \varphi(x, y, t)$ are basic unknown variables.

We introduce potential functions $\Phi_j(x, y, t)$ and $X_j(x, y, t)$ denoted by

$$u_x = \frac{\partial \Phi_1}{\partial x} + \frac{\partial \Phi_2}{\partial y}, \quad u_y = \frac{\partial \Phi_1}{\partial y} - \frac{\partial \Phi_2}{\partial x}, \quad \varphi = \frac{\partial X_1}{\partial y} - \frac{\partial X_2}{\partial x}. \tag{6}$$

Substituting Eq.(6) into Eq.(5), two equations in terms of $\Phi_j(x, y, t)$ and $X_j(x, y, t)$ can be derived as

$$A_j\frac{\partial^2 \Phi_j}{\partial x^2} + B_j\frac{\partial^2 \Phi_j}{\partial y^2} = C_j\ddot{\Phi}_j, \quad j = 1, 2, \tag{7a}$$

$$\varepsilon_{11}\frac{\partial^2 X_j}{\partial x^2} + \varepsilon_{33}\frac{\partial^2 X_j}{\partial y^2} = U_j\frac{\partial^2 \Phi_j}{\partial x^2} + V_j\frac{\partial^2 \Phi_j}{\partial y^2}, \quad j = 1, 2, \tag{7b}$$

where

$$A_1 = c_{11}(e_{33}\varepsilon_{11} - e_{15}\varepsilon_{33}) - (e_{15} + e_{31})[(2c_{44} + c_{13})\varepsilon_{11} + e_{15}(2e_{15} + e_{31})],$$
$$A_2 = (e_{33}\varepsilon_{11} - e_{15}\varepsilon_{33})(c_{11} - c_{13} - c_{44}) - (e_{15} + e_{31})(c_{44}\varepsilon_{33} + e_{15}e_{33}),$$
$$B_1 = (2c_{44} + c_{13})(e_{33}\varepsilon_{11} - e_{15}\varepsilon_{33}) - (e_{15} + e_{31})(c_{33}\varepsilon_{11} + e_{33}e_{15}),$$
$$B_2 = c_{44}(e_{33}\varepsilon_{11} - e_{15}\varepsilon_{33}) - (e_{15} + e_{31})[(c_{33} - c_{13} - c_{44})\varepsilon_{33} + (e_{33} - e_{15} - e_{31})e_{33}],$$
$$C_1 = [(e_{33} - e_{15} - e_{31})\varepsilon_{11} - e_{15}\varepsilon_{33}]\rho,$$
$$C_2 = [e_{33}\varepsilon_{11} - (2e_{15} + e_{31})\varepsilon_{33}]\rho,$$
$$U_1 = 2e_{15} + e_{31}, \quad V_1 = e_{33}, \quad U_2 = e_{15}, \quad V_2 = e_{33} - e_{15} - e_{31}.$$

For 2D problems in ordinary isotropic piezoelectric materials, the coefficients A_1, B_1 and C_1 in Eq.(7a) are all positive or negative, so are the coefficients A_2, B_2 and C_2. It is concluded that Eq.(7a) has the same form with the wave equation. Substituting Eq.(6) into Eqs.(1)–(3) the corresponding components of strain, electric field, stress and electric displacement are

$$\gamma_{xx} = \frac{\partial^2\Phi_1}{\partial x^2} + \frac{\partial^2\Phi_2}{\partial x\partial y}, \quad \gamma_{yy} = \frac{\partial^2\Phi_1}{\partial y^2} - \frac{\partial^2\Phi_2}{\partial x\partial y}, \quad \gamma_{xy} = 2\frac{\partial^2\Phi_1}{\partial x\partial y} + \frac{\partial^2\Phi_2}{\partial y^2} - \frac{\partial^2\Phi_2}{\partial x^2}; \tag{8}$$

$$E_x = -\frac{\partial^2 X_1}{\partial x\partial y} + \frac{\partial^2 X_2}{\partial x^2}, \quad E_y = -\frac{\partial^2 X_1}{\partial y^2} + \frac{\partial^2 X_2}{\partial x\partial y}, \tag{9}$$

$$\begin{cases} \sigma_{xx} = c_{11}(\frac{\partial^2\Phi_1}{\partial x^2} + \frac{\partial^2\Phi_2}{\partial x\partial y}) + c_{13}(\frac{\partial^2\Phi_1}{\partial y^2} - \frac{\partial^2\Phi_2}{\partial x\partial y}) + e_{31}(\frac{\partial^2 X_1}{\partial y^2} - \frac{\partial^2 X_2}{\partial x\partial y}), \\ \sigma_{yy} = c_{13}(\frac{\partial^2\Phi_1}{\partial x^2} + \frac{\partial^2\Phi_2}{\partial x\partial y}) + c_{33}(\frac{\partial^2\Phi_1}{\partial y^2} - \frac{\partial^2\Phi_2}{\partial x\partial y}) + e_{33}(\frac{\partial^2 X_1}{\partial y^2} - \frac{\partial^2 X_2}{\partial x\partial y}), \\ \sigma_{xy} = c_{44}(2\frac{\partial^2\Phi_1}{\partial x\partial y} + \frac{\partial^2\Phi_2}{\partial y^2} - \frac{\partial^2\Phi_2}{\partial x^2}) + e_{15}(\frac{\partial^2 X_1}{\partial x\partial y} - \frac{\partial^2 X_2}{\partial x^2}), \end{cases} \tag{10}$$

$$\begin{cases} D_x = e_{15}(2\frac{\partial^2\Phi_1}{\partial x\partial y} + \frac{\partial^2\Phi_2}{\partial y^2} - \frac{\partial^2\Phi_2}{\partial x^2}) - \varepsilon_{11}(\frac{\partial^2 X_1}{\partial x\partial y} - \frac{\partial^2 X_2}{\partial x^2}), \\ D_y = e_{31}(\frac{\partial^2\Phi_1}{\partial x^2} + \frac{\partial^2\Phi_2}{\partial x\partial y}) + e_{33}(\frac{\partial^2\Phi_1}{\partial y^2} - \frac{\partial^2\Phi_2}{\partial x\partial y}) + \varepsilon_{33}(\frac{\partial^2 X_1}{\partial y^2} - \frac{\partial^2 X_2}{\partial x\partial y}). \end{cases} \tag{11}$$

From what was discussed above, it is concluded that solving the 2D dynamic problem in the isotropic piezoelectric materials can be transformed into working for appropriate $\Phi_j(x,y,t)$ and $X_j(x,y,t)$ to satisfy Eq.(7). Then according to Eqs.(6), (8)–(11), the components of the coupled fields can be obtained. It is certain that, for different problems, the initial conditions and the boundary conditions still need to be considered. Therefore, the combination of Eqs.(7) and (6) becomes the so-called general solutions to the potential functions of the 2D dynamic problem in isotropic piezoelectric materials. Since $\Phi_j(x,y,t)$ and $X_j(x,y,t)$ are required to satisfy Eq.(7), the elastic displacement function $u_j = u_j(x,y,t)$ and the electric potential function $\varphi = \varphi(x,y,t)$ obtained from Eq.(6) must satisfy the equation group (5).

2 Coupled equations in problem of type I crack

Next, we will discuss the I-type crack problem in the piezoelectric material in terms of the application of the approach mentioned above. Several assumptions are made for this problem: (i) the x-axis is directed along the line of the crack; (ii)$L = 2a$; (iii) the center of the crack is coincident with the coordinate origin; (iv) the medium is free of load at infinity, and the impact load is imposed on the up surface as well as the down surface. Because of the symmetry, only a quarter of the plane needs to be considered. The boundary conditions can be written as

$$\sigma_{yy}(x,0,t) = -\sigma_0 H(t), \quad \sigma_{xy}(x,0,t) = 0, \qquad 0 < x < a,\ t > 0, \tag{12a}$$

$$D_y(x,0,t) = -D_0 H(t), \qquad 0 < x < a,\ t > 0, \tag{12b}$$

$$u_y(x,0,t) = 0, \quad \sigma_{xy}(x,0,t) = 0, \qquad x > a,\ t > 0, \tag{13a}$$

$$\varphi(x,0,t) = 0, \qquad x > a,\ t > 0, \tag{13b}$$

$$\sigma_{ij}(x,y,t) = 0, \qquad \text{at infinity},\ t > 0, \tag{14a}$$

$$D_i(x,y,t) = 0, \qquad \text{at infinity},\ t > 0, \tag{14b}$$

where $H(t)$ is Heaviside unit-step function. It is assumed that the initial conditions of the piezoelectric medium are

$$u_j(x,y,0) = \ddot{u}_j(x,y,0) = 0, \tag{15a}$$

$$\varphi(x,y,0)=\ddot{\varphi}(x,y,0)=0. \tag{15b}$$

For potential functions $\Phi_j(x,y,t)$ and $X_j(x,y,t)$, both Laplace transformations and their reverse transformations are made, while for the differential equation group (7) only Laplace transform is performed. According to the initial condition (15), we can derive

$$A_j\frac{\partial^2\Phi_j^*}{\partial x^2}+B_j\frac{\partial^2\Phi_j^*}{\partial y^2}=C_jp^2\Phi_j^*,\quad j=1,2, \tag{16a}$$

$$\varepsilon_{11}\frac{\partial^2X_j^*}{\partial x^2}+\varepsilon_{33}\frac{\partial^2X_j^*}{\partial y^2}=U_j\frac{\partial^2\Phi_j^*}{\partial x^2}+V_j\frac{\partial^2\Phi_j^*}{\partial y^2},\quad j=1,2. \tag{16b}$$

As the problem is symmetric and the electric potential function is a scalar one, the potential functions have following properties:

$$\Phi_1(x,y,t)=\Phi_1(-x,y,t),\quad \Phi_2(x,y,t)=-\Phi_2(-x,y,t);$$
$$X_1(x,y,t)=X_1(-x,y,t),\quad X_2(x,y,t)=-X_2(-x,y,t).$$

For the equation group (16), Fourier cosine transformation is made when $j=1$, while Fourier sine transform is made when $j=2$. Thus, the following equations can be obtained:

$$B_j\frac{\partial^2\tilde{\Phi}_j^*}{\partial y^2}-(A_js^2+C_jp^2)\tilde{\Phi}_j^*=0,\quad j=1,2, \tag{17a}$$

$$\varepsilon_{33}\frac{\partial^2\tilde{X}_j^*}{\partial y^2}-\varepsilon_{11}s^2\tilde{X}_j^*=\frac{(A_jV_j-B_jU_j)s^2+C_jV_jp^2}{B_j}\tilde{\Phi}_j^*,\quad j=1,2. \tag{17b}$$

By applying the boundary conditions at infinite, $\tilde{\Phi}_j^*(s,y,p)$ and $\tilde{X}_j^*(s,y,p)$ can be evaluated as follows:

$$\begin{cases}\tilde{\Phi}_j^*(s,y,p)=F_j(s,p)\exp(-\omega_jy),\quad j=1,2,\\ \tilde{X}_j^*(s,y,p)=G(s,p)\exp(-\lambda y)+H_j(s,p)F_j(s,p)\exp(-\omega_jy),\quad j=1,2,\end{cases} \tag{18}$$

where

$$\omega_j=\sqrt{(A_js^2+C_jp^2)/B_j},\quad \lambda=\sqrt{\varepsilon_{11}s^2/\varepsilon_{33}},$$
$$H_j(s,p)=(V_j\omega_j^2-U_js^2)/(\varepsilon_{33}\omega_j^2-\varepsilon_{11}s^2).$$

Fourier cosine reverse transformations are made for the image functions $\tilde{\Phi}_1^*(s,y,p)$ and $\tilde{X}_1^*(s,y,p)$ in Eq.(18), and at the same time, Fourier sine reverse transformations are made for the image functions $\tilde{\Phi}_2^*(s,y,p)$ and $\tilde{X}_2^*(s,y,p)$. After these steps, we have

$$\begin{cases}\Phi_1^*(x,y,p)=\dfrac{2}{\pi}\displaystyle\int_0^\infty F_1(s,p)\exp(-\omega_1y)\cos(sx)ds,\\ \Phi_2^*(x,y,p)=\dfrac{2}{\pi}\displaystyle\int_0^\infty F_2(s,p)\exp(-\omega_2y)\sin(sx)ds,\end{cases} \tag{19}$$

$$\begin{cases}X_1^*(x,y,p)=\dfrac{2}{\pi}\displaystyle\int_0^\infty[G(s,p)\exp(-\lambda y)+H_1(s,p)F_1(s,p)\exp(-\omega_1y)]\cos(sx)ds,\\ X_2^*(x,y,p)=\dfrac{2}{\pi}\displaystyle\int_0^\infty[G(s,p)\exp(-\lambda y)+H_2(s,p)F_2(s,p)\exp(-\omega_2y)]\sin(sx)ds.\end{cases} \tag{20}$$

Making Laplace transformations for the boundary conditions (12) and (13), we can obtain

$$\sigma^*_{yy}(x,0,p) = -\sigma_0/p, \quad \sigma^*_{xy}(x,0,p) = 0, \qquad 0 < x < a, \tag{21a}$$

$$D^*_y(x,0,p) = -D_0/p, \qquad 0 < x < a, \tag{21b}$$

$$u^*_y(x,0,p) = 0, \quad \sigma^*_{xy}(x,0,p) = 0, \qquad x > a, \tag{22a}$$

$$\varphi^*(x,0,p) = 0, \qquad x > a. \tag{22b}$$

We make Laplace transformations for Eq.(6) and Eqs.(8)–(11), and then substitute Eqs.(19) and (20) into them. By applying the symmetry and the boundary condition $\sigma^*_{xy}(x,0,p) = 0$, $-\infty < x < +\infty$, we can obtain

$$G(s,p) = -\frac{\omega_1 H_1(s,p)F_1(s,p) + sH_2(s,p)F_2(s,p)}{\lambda + s} - \frac{c_{44}[2s\omega_1 F_1(s,p) + (\omega_2^2 + s^2)F_2(s,p)]}{e_{15}s(\lambda + s)}. \tag{23}$$

According to the boundary conditions $u^*_y(x,0,p) = 0, \varphi^*(x,0,p) = 0, x > a$, we can obtain

$$\int_0^\infty [\omega_1 F_1(s,p) + sF_2(s,p)]\cos(sx)ds = 0, \quad x > a, \tag{24a}$$

$$\int_0^\infty [\lambda G(s,p) + \omega_1 H_1(s,p)F_1(s,p) + sG(s,p) + sH_2(s,p)F_2(s,p)]\cos(sx)ds, \quad x > a. \tag{24b}$$

Substituting Eq.(23) into Eq.(24b) and comparing with Eq.(24a), we can obtain

$$\int_0^\infty [\omega_1 F_1(s,p) + \omega_2^2 s^{-1} F_2(s,p)]\cos(sx)ds = 0, \quad x > a. \tag{25}$$

According to the boundary conditions $\sigma^*_{yy}(x,0,p) = -\sigma_0/p, D^*_y(x,0,p) = -D_0/p, 0 < x < a$, and Eq.(23), we can get

$$\begin{cases} \displaystyle\int_0^\infty [T_{11}(s,p)F_1(s,p) + T_{12}(s,p)F_2(s,p)]\cos(sx)ds = \frac{\pi\sigma_0}{2p}, & 0 < x < a, \\ \displaystyle\int_0^\infty [T_{21}(s,p)F_1(s,p) + T_{22}(s,p)F_2(s,p)]\cos(sx)ds = \frac{\pi D_0}{2p}, & 0 < x < a, \end{cases} \tag{26}$$

where

$$T_{11}(s,p) = c_{13}s^2 - c_{33}\omega_1^2 - e_{33}\omega_1(\omega_1 - \lambda)H_1(s,p) + 2\sqrt{\varepsilon_{11}\varepsilon_{33}^{-1}}c_{44}e_{33}e_{15}^{-1}s\omega_1,$$

$$T_{12}(s,p) = (c_{13} - c_{33})s\omega_2 - e_{33}s(\omega_2 - \lambda)H_2(s,p) + \sqrt{\varepsilon_{11}\varepsilon_{33}^{-1}}c_{44}e_{33}e_{15}^{-1}(\omega_2^2 + s^2),$$

$$T_{21}(s,p) = e_{31}s^2 - e_{33}\omega_1^2 + \varepsilon_{33}\omega_1(\omega_1 - \lambda)H_1(s,p) - 2\sqrt{\varepsilon_{11}\varepsilon_{33}}c_{44}e_{15}^{-1}s\omega_1,$$

$$T_{22}(s,p) = (e_{31} - e_{33})s\omega_2 + \varepsilon_{33}s(\omega_2 - \lambda)H_2(s,p) - \sqrt{\varepsilon_{11}\varepsilon_{33}}c_{44}e_{15}^{-1}(\omega_2^2 + s^2).$$

Consequently, solving such a problem can be transformed into searching for $F_j(s,p)$ to satisfy the coupled integral equations (24a), (25) and (26).

3 Solutions of dual integral equations

We assume that

$$\begin{cases} g_1(s,p) = \omega_1 F_1(s,p) + sF_2(s,p), \\ g_2(s,p) = \omega_1 F_1(s,p) + \omega_2^2 s^{-1} F_2(s,p). \end{cases} \tag{27}$$

Then, we can obtain

$$\begin{cases} F_1(s,p) = \dfrac{s^2 g_2(s,p) - \omega_2^2 g_1(s,p)}{\omega_1(s^2-\omega_2^2)}, \\ F_2(s,p) = \dfrac{s g_1(s,p) - s g_2(s,p)}{s^2-\omega_2^2}. \end{cases} \tag{28}$$

Substituting $\cos(sx) = \sqrt{\pi xs/2}\mathrm{J}_{-1/2}(sx)$, Eqs.(27) and (28) into Eqs.(24a), (25) and (26), the following dual integral equations can be derived:

$$\begin{cases} \displaystyle\int_0^\infty \sqrt{s} g_1(s,p)\mathrm{J}_{-1/2}(sx)ds = 0, \quad x > a, \\ \displaystyle\int_0^\infty \sqrt{s} g_2(s,p)\mathrm{J}_{-1/2}(sx)ds = 0, \quad x > a, \end{cases} \tag{29}$$

$$\begin{cases} \displaystyle\int_0^\infty s^2[m_{11}g_1(s,p) + m_{12}g_2(s,p)]\mathrm{J}_{-1/2}ds = T_{10}(x), \quad 0 < x < a, \\ \displaystyle\int_0^\infty s^2[m_{21}g_1(s,p) + m_{22}g_2(s,p)]\mathrm{J}_{-1/2}ds = T_{20}(x), \quad 0 < x < a, \end{cases} \tag{30}$$

where $\mathrm{J}_\nu(sx)$ is the ν-order Bessel function, while

$$T_{10}(x) = \frac{\sqrt{\pi}\sigma_0}{\sqrt{2xp}}, \quad T_{20}(x) = \frac{\sqrt{\pi}D_0}{\sqrt{2xp}},$$

$$m_{11} = m_{11}(s,p) = \frac{T_{12}(s,p)\omega_1 s - T_{11}(s,p)\omega_2^2}{\omega_1 s(s^2-\omega_2^2)\sqrt{s}}, \quad m_{12}(s,p) = \frac{T_{11}(s,p)s - T_{12}(s,p)\omega_1}{\omega_1(s^2-\omega_2^2)\sqrt{s}},$$

$$m_{21}(s,p) = \frac{T_{22}(s,p)\omega_1 s - T_{21}(s,p)\omega_2^2}{\omega_1 s(s^2-\omega_2^2)\sqrt{s}}, \quad m_{22}(s,p) = \frac{T_{21}(s,p)s - T_{22}(s,p)\omega_1}{\omega_1(s^2-\omega_2^2)\sqrt{s}},$$

$t_1(x)$ and $t_2(x)$ are expressed as

$$\begin{cases} t_1(x) = T_{10}(x) + \displaystyle\int_0^\infty s^2[((\chi(C_1/A_1)^{1/4}\sqrt{p})^{-1} - m_{11})g_1(s,p) - m_{12}g_2(s,p)]\mathrm{J}_{-1/2}(sx)ds, \\ t_2(x) = T_{20}(x) + \displaystyle\int_0^\infty s^2[((\chi(C_2/A_2)^{1/4}\sqrt{p})^{-1} - m_{22})g_2(s,p) - m_{21}g_1(s,p)]\mathrm{J}_{-1/2}(sx)ds, \end{cases} \tag{31}$$

where χ must be a positive constant. Substituting Eq.(31) into Eq.(30), we can obtain

$$\begin{cases} \displaystyle\int_0^\infty s^2 g_1(s,p)\mathrm{J}_{-1/2}(sx)dx = t_1(x)\chi(C_1/A_1)^{1/4}\sqrt{p}, \quad 0 < x < a, \\ \displaystyle\int_0^\infty s^2 g_2(s,p)\mathrm{J}_{-1/2}(sx)dx = t_2(x)\chi(C_2/A_2)^{1/4}\sqrt{p}, \quad 0 < x < a. \end{cases} \tag{32}$$

The unknown functions $g_1(s,p)$ and $g_2(s,p)$ are expressed as the integral of the new unknown functions $\phi_1(\xi,p)$ and $\phi_2(\xi,p)$ as follows:

$$g_j(s,p) = \frac{\pi a^2}{2p}s^{-1/4}\int_0^1 \sqrt{\xi}\phi_j(\xi,p)\mathrm{J}_{1/4}(sa\xi)d\xi, \tag{33}$$

where $\phi_1(\xi,p)$ and $\phi_2(\xi,p)$ must satisfy

$$\lim_{\xi\to 0^+} \xi^{-1}\phi_j(\xi,p) = 0. \tag{34}$$

The disconnected integral formulas of Bessel function are as follows[8]:

$$\int_0^\infty \mathrm{J}_\lambda(r\xi)\mathrm{J}_\mu(b\xi)\xi^{1+\mu-\lambda}d\xi = 0, \quad 0<r<b, \tag{35a}$$

$$\int_0^\infty \mathrm{J}_\lambda(r\xi)\mathrm{J}_\mu(b\xi)\xi^{1+\mu-\lambda}d\xi = \frac{b^\mu(r^2-b^2)^{\lambda-\mu-1}}{2^{\lambda-\mu-1}r^\lambda\Gamma(\lambda-\mu)}, \quad 0<b<r, \tag{35b}$$

where $\Gamma(z)$ is Γ function and $\lambda>\mu>-1$ must be satisfied. Substituting Eq.(33) into Eq.(29) and changing the integral order, we can obtain

$$\int_0^\infty \sqrt{s}g_j(s,p)\mathrm{J}_{-1/2}(sx)ds = \frac{\pi a^2}{2p}\int_0^1 \sqrt{\xi}\phi_j(\xi,p)\int_0^\infty s^{1/4}\mathrm{J}_{1/4}(sa\xi)\mathrm{J}_{-1/2}(sx)dsd\xi.$$

According to the disconnected integral formulas of Eq.(35), the equation above equals zero. So Eq.(29) can be satisfied automatically. By applying the differential formula of the Bessel function[9],

$$\frac{d}{dz}[z^{-\nu}\mathrm{J}_\nu(z)] = -z^{-\nu}\mathrm{J}_{\nu+1}(z),$$

the subsection integral is made for the right side of Eq.(33). Because of Eq.(34), the following equation can be derived:

$$g_j(s,p) = -\frac{\pi a}{2s^{5/4}p}\{\phi_j(1,p)\mathrm{J}_{-3/4}(as) - \int_0^1 (as\xi)^{3/4}\mathrm{J}_{-3/4}(as\xi)\frac{d}{d\xi}[\frac{\sqrt{\xi}\phi_j(\xi,p)}{(as\xi)^{3/4}}]d\xi\}.$$

Substituting the equation above into Eq.(32) and applying the disconnected formulas of Eq.(35), the Abel-type integral equation is derived as

$$t_j(x) = \frac{(aA_j)^{1/4}\pi\sqrt{x}}{(2C_j)^{1/4}\chi p^{3/2}\Gamma(1/4)}\int_0^{x/a}\frac{1}{(x^2-a^2\xi^2)^{3/4}}\frac{d}{d\xi}[\xi^{-1/4}\phi_j(\xi,p)]d\xi \quad (0<\xi<x/a). \tag{36}$$

By using the Abel-type integral equation and its reverse transformation formula[10] and specifying

$$h(x) = \frac{(2C_j)^{1/4}\chi p^{3/2}\Gamma(1/4)}{(aA_j)^{1/4}\pi\sqrt{x}}t_j(x) = \int_0^{x/a}\frac{1}{(x^2-a^2\xi^2)^{3/4}}\frac{d}{d\xi}[\xi^{-1/4}\phi_j(\xi,p)]d\xi,$$

$r(\xi)$ can be expressed as

$$r(\xi) = \frac{-2\sin(-3\pi/4)}{\pi}\frac{d}{d\xi}\int_0^{a\xi}(a^2\xi^2-x^2)^{-1/4}xh(x)dx = \frac{d}{d\xi}[\xi^{-1/4}\phi_j(\xi,p)].$$

Consequently, we can obtain

$$\phi_j(\xi,p) = \frac{(\xi C_j)^{1/4}2^{3/4}\chi p^{3/2}\Gamma(1/2)}{(aA_j)^{1/4}\pi^2}\int_0^{a\xi}\frac{\sqrt{x}t_j(x)}{(a^2\xi^2-x^2)^{1/4}}dx. \tag{37}$$

Substituting Eq.(31) into Eq.(37) and specifying

$$\beta_1 = [(\xi C_j)^{1/4}2^{3/4}\chi p^{3/2}\Gamma(1/2)]/[(aA_j)^{1/4}\pi^2],$$

we can obtain

$$\phi_j(\xi,p) = \beta_1\int_0^{a\xi}\frac{\sqrt{x}[T_{10}(x)+\int_0^\infty s^2[((\chi(C_1/A_1)^{1/4}\sqrt{p})^{-1}-m_{11})g_1-m_{12}g_2]\mathrm{J}_{-1/2}(sx)ds]}{(a^2\xi^2-x^2)^{1/4}}dx. \tag{38a}$$

Making the integral and subsection integral for the right side of Eq.(38a) and using identical equation

$$\int_0^t x^{\nu+1}\mathrm{J}_\nu(sx)\frac{dx}{(t^2-x^2)^{1-\alpha}} = 2^{\alpha-1}\Gamma(\alpha)s^{-\alpha}t^{\alpha+\nu}\mathrm{J}_{\alpha+\nu}(st),\quad \Gamma(1/2)=\pi, \tag{38b}$$

second type Fredholm integral equations can be obtained as follows:

$$\begin{cases} \dfrac{A_1^{1/4}}{\chi\sqrt{p}C_1^{1/4}}\phi_1(\xi,p) + \displaystyle\int_0^1 [K_{11}\phi_1(\eta,p) + K_{12}\phi_2(\eta,p)]d\eta = 0.454a^{1/4}\xi^{3/4}\sigma_0, \\ \dfrac{A_2^{1/4}}{\chi\sqrt{p}C_2^{1/4}}\phi_2(\xi,p) + \displaystyle\int_0^1 [K_{21}\phi_1(\eta,p) + K_{22}\phi_2(\eta,p)]d\eta = 0.454a^{1/4}\xi^{3/4}D_0, \end{cases} \tag{39}$$

where the integrants K_{ij} of the integral equations are

$$K_{11}(\xi,\eta,p) = 0.489a^2\sqrt{\xi\eta}\int_0^\infty s[m_{11}(s,p) - (\chi(C_1/A_1)^{1/4}\sqrt{p})^{-1}]\mathrm{J}_{1/4}(sa\xi)\mathrm{J}_{1/4}(sa\eta)ds,$$

$$K_{12}(\xi,\eta,p) = 0.489a^2\sqrt{\xi\eta}\int_0^\infty sm_{12}(s,p)\mathrm{J}_{1/4}(sa\xi)\mathrm{J}_{1/4}(sa\eta)ds,$$

$$K_{21}(\xi,\eta,p) = 0.489a^2\sqrt{\xi\eta}\int_0^\infty sm_{21}(s,p)\mathrm{J}_{1/4}(sa\xi)\mathrm{J}_{1/4}(sa\eta)ds,$$

$$K_{22}(\xi,\eta,p) = 0.489a^2\sqrt{\xi\eta}\int_0^\infty s[m_{22}(s,p) - (\chi(C_2/A_2)^{1/4}\sqrt{p})^{-1}]\mathrm{J}_{1/4}(sa\xi)\mathrm{J}_{1/4}(sa\eta)ds.$$

Changing Eq.(39) into algebraic equations and solving them by Matlab program, functions can be derived. The detail solution process is shown in Ref.[11].

4 Conclusion

The reference [12] has indicated the existing mathematics problem in the integral transform of the dynamic crack for ordinary materials. This paper successfully resolved the problem, and at the same time, developed a kind of approach to solve such problems.

In this paper, the basic equations of dynamic problems in piezoelectric materials are expressed by three introduced potential functions, and the differential equations are solved through Laplace transformation and Fourier transformation, which are based on mature and meticulous mathematical theories and methods. Coupled integral equations of the dynamic problem are established for the type I crack under the impact load in piezoelectric materials, and the approach to changing coupled integral equations into second type Fredholm ones is investigated in detail. The approach presented in this paper was determined to be feasible and expected to be used to study the dynamic crack problem in piezoelectric materials.

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